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Value sharing results for shifts of meromorphic functions, and sufficient conditions for periodicity ☆

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ABSTRACT

This research is a continuation of a recent paper due to the first four authors. Shared value problems related to a meromorphic function $f(z)$ and its shift $f(z+c)$, where $c \in \mathbb{C}$, are studied. It is shown, for instance, that if $f(z)$ is of finite order and shares two values CM and one value IM with its shift $f(z+c)$, then f is a periodic function with period c. The assumption on the order of f can be dropped if f shares two shifts in different directions, leading to a new way of characterizing elliptic functions. The research findings also include an analogue for shifts of a well-known conjecture by Brück concerning the value sharing of an entire function f with its derivative f'.

1. Introduction

We assume that the reader is familiar with the elementary Nevanlinna theory, see, e.g., [5,12,15,19]. Meromorphic functions are always non-constant, unless otherwise specified. As for the standard notation in the uniqueness theory of meromorphic functions, suppose that f, g are meromorphic and $a \in \widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, resp. a is a small meromorphic function in the usual Nevanlinna theory sense. Denoting by $E(a, f)$ the set of those points $z \in \mathbb{C}$ where $f(z) = a$, resp. $f(z) = a(z)$, we say that f, g share a IM (ignoring multiplicities), if $E(a, f) = E(a, g)$. Provided that $E(a, f) = E(a, g)$ and the multiplicities of the zeros of $f(z) - a$ and $g(z) - a$ are the same at each $z \in \mathbb{C}$, then f, g share a CM (counting multiplicities).

The classical results in the uniqueness theory of meromorphic functions are the five-point, resp. four-point, theorems due to Nevanlinna [17]: If two meromorphic functions f, g share five distinct values in the extended complex plane IM, then $f \equiv g$. Similarly, if two meromorphic functions f, g share four distinct values in the extended complex plane CM, then $f \equiv T \circ g$, where T is a Möbius transformation. The assumption 4 CM in the four-point theorem has been improved to 2 CM + 2 IM by Gundersen [7]. It is well known that 4 CM cannot be improved to 4 IM [6], while 1 CM + 3 IM remains an open problem.

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We recall a well-known conjecture by Brück [2]: Let f be an entire function such that the hyper-order

$$\rho_2(f) := \limsup_{r\to\infty} \frac{\log\log T(r,f)}{\log r}$$

is not a positive integer or infinity. If f and f' share one finite value a CM, then

$$\frac{f'-a}{f-a} = \tau \tag{1}$$

for some non-zero constant τ. Brück's conjecture has been verified in the special cases when $a = 0$ [2] or when f is of finite order [8]. Examples in [2] show that the conjecture does not hold if $\rho_2(f)$ is either a positive integer or infinity. Moreover, an example in [8] shows that the word "entire" cannot be replaced with the word "meromorphic". As for the extensive theory of uniqueness of meromorphic functions, see [19].

In a recent paper [13], the first four authors started to consider the uniqueness of meromorphic functions sharing values with their shifts. The background for these considerations lies in the recent interest of studying Nevanlinna theory with respect to difference operators, see, e.g., the papers [9,10] by Halburd and Korhonen and [3] by Chiang and Feng. Further, shared value problems of meromorphic functions with their shifts naturally appear by looking at shared values of f and $f \circ p$, as is seen next.

Theorem A. *(See [13, Theorem 1.5].) Let f and p be entire functions. If f is transcendental and shares two distinct values $a_1, a_2 \in \mathbb{C}$ IM with $f \circ p$, then $p(z) = \alpha z + \beta$ for some constants $\alpha, \beta \in \mathbb{C}$ with $|\alpha| = 1$. The same conclusion holds if f is a polynomial and IM is replaced by CM.*

We specify the notion of small functions as follows: Given a meromorphic function f, the family of all meromorphic functions ω such that $T(r,\omega) = o(T(r,f))$, where $r \to \infty$ outside of a possible exceptional set of finite logarithmic measure, is denoted by $\mathcal{S}(f)$. For convenience, we also include all constant functions in $\mathcal{S}(f)$. Moreover, let $\widehat{\mathcal{S}}(f) = \mathcal{S}(f) \cup \{\infty\}$. The two key results in [13] now read as follows.

Theorem B. *(See [13, Theorem 2.1(a)].) Let f be a meromorphic function of finite order, and let $c \in \mathbb{C}$. If $f(z)$ and $f(z+c)$ share three distinct periodic functions $a_1, a_2, a_3 \in \widehat{\mathcal{S}}(f)$ with period c CM, then $f(z) = f(z+c)$ for all $z \in \mathbb{C}$.*

Theorem C. *(See [13, Theorem 2.3].) Let f be a meromorphic function, and let $c_1, c_2 \in \mathbb{C}$ be linearly independent over the real numbers. If $f(z)$, $f(z+c_1)$ and $f(z+c_2)$ share three distinct values $a_1, a_2, a_3 \in \widehat{\mathbb{C}}$ CM, then f is an elliptic function with periods c_1 and c_2.*

The following counterexample from [13] shows that the assumption on the finiteness of the order of growth in Theorem B cannot be dropped. Let $c \in \mathbb{C} \setminus \{0\}$, and let $f(z) = \exp(\sin(\pi z/c))$. Clearly, f is of infinite order of growth, and $f(z)$ and $f(z+c)$ share 0, 1 and ∞ CM, yet the functions $f(z)$ and $f(z+c)$ are not the same.

In Section 2 we prove a shifted analogue of Brück's conjecture valid for meromorphic functions, see Theorem 1. The considerations in Sections 3–5 below are devoted to improving Theorems B and C by relaxing the sharing conditions. In particular, Theorem 2 shows that 3 CM in Theorem B can be replaced with 2 CM + 1 IM. It remains open, whether this could be improved to 1 CM + 2 IM, or even to 3 IM. Moreover, Theorem C, a characterization of elliptic functions, similarly improves from 3 CM to 2 CM + 1 IM, see Theorem 10. In Theorems 6–8 and 12 we proceed to reduce the number of the shared small periodic functions, assuming that the meromorphic function f under consideration, or in fact a simple transformation of f, is close to an entire function in the sense that a certain deficiency condition applies. Moreover, we discuss the generally open 3 IM situation by introducing Theorem 14.

In the final Section 6, we prove variants of Theorem A. Theorem 16 is a meromorphic analogue of Theorem A, being a slight improvement at the same time. Theorem 18 presents a special case of Theorem A, assuming that one of the values shared by f and $f \circ p$ is a Picard value.

In addition to basic results from Nevanlinna theory, a difference analogue of the lemma on the logarithmic derivative from [3,9,10] takes a key role in the proofs below. For the convenience of the reader, this lemma and a couple of other auxiliary results from the difference variant of value distribution theory will be recalled whenever needed.

2. An analogue of Brück's conjecture

The next result is a shifted analogue of Brück's conjecture valid for meromorphic functions.

Theorem 1. *Let f be a meromorphic function of order of growth*

$$\rho(f) := \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r} < 2, \tag{2}$$

and let $c \in \mathbb{C}$. If $f(z)$ and $f(z+c)$ share the values $a \in \mathbb{C}$ and ∞ CM, then

$$\frac{f(z+c)-a}{f(z)-a}=\tau \tag{3}$$

for some constant τ.

To illustrate the necessity of the growth restriction (2), let $f(z)=e^{z^2}+1$ and $c\in\mathbb{C}$. Then the functions $f(z)$ and $f(z+c)$ share the values 1 and ∞ CM, and yet

$$\frac{f(z+c)-1}{f(z)-1}=e^{2cz+c^2}\not\equiv \text{constant}.$$

Since $\rho(f)=2$, this counterexample shows that $\rho(f)<2$ cannot be relaxed to $\rho(f)\leqslant 2$.

We write (3) as a first-order linear difference equation

$$f(z+c)-\tau f(z)=a(1-\tau),$$

whose solutions of order <2 can be written as $f(z)=d(z)\exp(\frac{\operatorname{Log}\tau}{c}z)+a$, where Log denotes the principal branch of the logarithm, and d is a periodic function with period c such that $\rho(d)\in[0,2)$. In particular, if $\tau=1$, then f is a periodic function with period c. Note that for any $\sigma\in[1,\infty)$ there exists a prime periodic entire function h of order $\rho(h)=\sigma$ by [18, Theorem 1]. Analogously, if (1) is to be considered as a first-order linear differential equation, then its general solution can be written as $f(z)=d\exp(\tau z)+\frac{a(\tau-1)}{\tau}$, $d\in\mathbb{C}$, which is a periodic entire function with period $\frac{2\pi i}{\tau}$ such that $\rho(f)=1$.

To prove Theorem 1, we need the following result on quotients of shifts.

Theorem D. *(See [9, Theorem 2.1], [10, Corollary 2.2].) Let f be a meromorphic function of finite order, and let $c\in\mathbb{C}$ and $\delta\in(0,1)$. Then*

$$m\left(r,\frac{f(z+c)}{f(z)}\right)+m\left(r,\frac{f(z)}{f(z+c)}\right)=o\left(\frac{T(r,f)}{r^{\delta}}\right)$$

for all r outside of a possible exceptional set E with finite logarithmic measure.

Proof of Theorem 1. It follows by the assumptions that

$$\frac{f(z+c)-a}{f(z)-a}=e^{Q(z)},$$

where Q is a polynomial of degree at most one. Theorem D yields

$$T\big(r,e^{Q(z)}\big)=m\big(r,e^{Q(z)}\big)=o\big(r^{\rho(f)+\varepsilon-\delta}\big)$$

for any $\varepsilon>0$ and $\delta\in(0,1)$. Since $\rho(f)<2$, we deduce that $Q(z)$ must be a constant. $\square$

Under the assumptions of Theorem D, it is evident that

$$m\left(r,\frac{f(z+c)}{f(z)}\right)+m\left(r,\frac{f(z)}{f(z+c)}\right)=S(r,f).$$

This fact will be used later on whenever referring to Theorem D. Chiang and Feng have obtained similar estimates for the logarithmic differences in [3], and this work is independent from [9,10].

3. Improvements of Theorem B

Next we show that 3 CM in Theorem B can be replaced with 2 CM + 1 IM.

Theorem 2. *Let f be a meromorphic function of finite order, let $c\in\mathbb{C}$, and let $a_1,a_2,a_3\in\widehat{\mathcal{S}}(f)$ be three distinct periodic functions with period c. If $f(z)$ and $f(z+c)$ share a_1,a_2 CM and a_3 IM, then $f(z)=f(z+c)$ for all $z\in\mathbb{C}$.*

Theorem 2 has the following two immediate consequences.

Corollary 3. *Let f be an entire function of finite order, let $c\in\mathbb{C}$, and let $a,b\in\mathcal{S}(f)$ be two distinct periodic functions with period c. If $f(z)$ and $f(z+c)$ share a CM and b IM, then $f(z)=f(z+c)$ for all $z\in\mathbb{C}$.*

Corollary 4. *Let $c\in\mathbb{C}$ and $f=e^{Q}$, where Q is a non-constant polynomial such that $f(z)-f(z+c)\not\equiv 0$. Then there does not exist a periodic function $a\in\mathcal{S}(f)\setminus\{0\}$ with period c such that $f(z)$ and $f(z+c)$ share a IM.*

To prove Theorem 2, we need the following result.

Theorem E. *(See [14, Theorem 4], [19, Theorem 5.3].) Let f_1, f_2, f_3, f_4 be meromorphic functions, and let $a_1, a_2, a_3 \in \widehat{\mathbb{C}}$ be three distinct points. If the functions f_1, f_2, f_3, f_4 share a_1, a_2 CM and a_3 IM, then at least two of f_1, f_2, f_3, f_4 are the same.*

Proof of Theorem 2. (1) Suppose first that $a_1, a_2, a_3 \in \mathcal{S}(f)$. Denote

$$g(z) = \frac{f(z) - a_1(z)}{f(z) - a_2(z)} \cdot \frac{a_3(z) - a_2(z)}{a_3(z) - a_1(z)}. \tag{4}$$

Then

$$g(z+c) = \frac{f(z+c) - a_1(z)}{f(z+c) - a_2(z)} \cdot \frac{a_3(z) - a_2(z)}{a_3(z) - a_1(z)}.$$

It suffices to show that $g(z) = g(z+c)$ for all $z \in \mathbb{C}$. Since now $g(z)$ and $g(z+c)$ share $0, \infty$ CM, and since g is of finite order, it follows that

$$\frac{g(z+c)}{g(z)} = e^{Q(z)},$$

where Q is a polynomial. Moreover, we conclude that the functions $g(z)$, $g(z+c)$, $g(z+2c)$, $g(z+3c)$ share $0, \infty$ CM and 1 IM. By Theorem E, at least two of these functions are the same. It suffices to consider the cases $g(z) \equiv g(z+2c)$ and $g(z) \equiv g(z+3c)$.

Suppose that $g(z) = g(z+2c)$ for all $z \in \mathbb{C}$. Then

$$1 = \frac{g(z+2c)}{g(z+c)} \cdot \frac{g(z+c)}{g(z)} = e^{Q(z+c)} \cdot e^{Q(z)}$$

for all $z \in \mathbb{C}$. This gives $Q(z) + Q(z+c) = 2n\pi i$ for some $n \in \mathbb{Z}$ and for all $z \in \mathbb{C}$. By writing $Q(z) = C_k z^k + C_{k-1} z^{k-1} + \cdots + C_0$, we have

$$C_k\big((z+c)^k + z^k\big) + C_{k-1}\big((z+c)^{k-1} + z^{k-1}\big) + \cdots + 2C_0 = 2n\pi i$$

for all $z \in \mathbb{C}$. Since the expressions $(z+c)^j + z^j$, $j = 1, \ldots, k$, are linearly independent, it follows that $C_1 = \cdots = C_k = 0$, and hence $Q(z) \equiv n\pi i$. If n is even, then $e^{Q(z)} \equiv 1$, and we are done. Suppose then that n is odd. Then $e^{Q(z)} \equiv -1$, that is, $g(z) = -g(z+c)$ for all $z \in \mathbb{C}$. If there exists a point $z_0 \in \mathbb{C}$ such that $g(z_0) = 1$, then also $g(z_0+c) = 1$, which is a contradiction with $g(z_0) = -g(z_0+c)$. Hence 1 must be a Picard value of $g(z)$ and of $g(z+c)$, and so the functions $g(z)$ and $g(z+c)$ share $0, 1, \infty$ CM. The assertion now follows by Theorem B.

Suppose then that $g(z) = g(z+3c)$ for all $z \in \mathbb{C}$. Then

$$1 = \frac{g(z+3c)}{g(z+2c)} \cdot \frac{g(z+2c)}{g(z+c)} \cdot \frac{g(z+c)}{g(z)} = e^{Q(z+2c)} \cdot e^{Q(z+c)} \cdot e^{Q(z)}$$

for all $z \in \mathbb{C}$. This gives $Q(z) + Q(z+c) + Q(z+2c) = 2n\pi i$ for some $n \in \mathbb{Z}$ and for all $z \in \mathbb{C}$, so that $Q(z) \equiv \frac{2}{3}n\pi i$. We have three possibilities: $n \equiv 0 \pmod 3$, $n \equiv 1 \pmod 3$ or $n \equiv 2 \pmod 3$. In the first case $e^{Q(z)} \equiv 1$, and we are done. In the two remaining cases 1 must be a Picard value of $g(z)$ and of $g(z+c)$, or else we arrive at a contradiction.

(2) Suppose then that $a_1 = \infty$, while $a_2, a_3 \in \mathcal{S}(f)$. Let $d \in \mathbb{C} \setminus \{a_2, a_3\}$. Denote $h(z) = 1/(f(z) - d)$, $b_2(z) = 1/(a_2(z) - d)$ and $b_3(z) = 1/(a_3(z) - d)$. Then $b_2, b_3 \in \mathcal{S}(h)$ are two distinct periodic meromorphic functions with period c. Moreover, the functions $h(z)$ and $h(z+c)$ share $0, b_2$ CM and b_3 IM. Part (1) implies that $h(z) = h(z+c)$ for all $z \in \mathbb{C}$, from which the assertion follows. The cases $a_2 = \infty$ and $a_3 = \infty$ are dealt with analogously. The fact whether the value ∞ is shared CM or IM plays no significant role in this reasoning. □

If 2 CM + 1 IM is replaced with 2 CM in Theorem 2, then an additional deficiency condition needs to be introduced as follows.

Theorem F. *(See [13, Theorem 2.1(b), (c)].) Let f be a meromorphic function of finite order, let $c \in \mathbb{C}$, and let $a_1, a_2, a_3 \in \widehat{\mathcal{S}}(f)$ be three distinct periodic functions with period c. If $f(z)$ and $f(z+c)$ share a_1, a_3 CM, and if*

$$\limsup_{r \to \infty} \frac{N(r, \frac{1}{f-a_2})}{T(r,f)} < 1, \tag{5}$$

then $f(z) = f(z+c)$ for all $z \in \mathbb{C}$.

The following lemma on the growth of non-decreasing real-valued functions will be needed in proving further refinements of Theorem 2.

Lemma G. *(See [11, Lemma 2.1].) Let $T:(0,+\infty)\to(0,+\infty)$ be a non-decreasing continuous function, $s>0$, $\alpha<1$, and let $F\subset\mathbb{R}_+$ be the set of all r such that*

$$T(r)\leqslant \alpha T(r+s).$$

If the logarithmic measure of F is infinite, then

$$\limsup_{r\to\infty}\frac{\log T(r)}{\log r}=\infty.$$

In order to prove a 1 CM + 1 IM version of Theorem F, the following version of the second main theorem is required.

Theorem 5. *Let f be a meromorphic function, let $\varepsilon>0$, and let a_1,a_2,a_3 be pairwise distinct meromorphic functions such that $a_1,a_2\in\mathcal{S}(f)$, and*

$$T(r,a_3)\leqslant \nu T(r,f)+S(r,f) \tag{6}$$

for some $\nu\in[0,1/3)$. Then

$$(1-3\nu-\varepsilon)T(r,f)\leqslant\sum_{j=1}^{3}\overline{N}\left(r,\frac{1}{f-a_j}\right)+S(r,f).$$

Proof. We apply the method of proof of the second main theorem for three small target functions [12, Theorem 2.5]. By defining $g(z)$ as in (4), the (usual) second main theorem yields

$$\begin{aligned}T(r,g)&\leqslant\overline{N}(r,g)+\overline{N}\left(r,\frac{1}{g}\right)+\overline{N}\left(r,\frac{1}{g-1}\right)+S(r,g)\\&\leqslant\sum_{j=1}^{3}\overline{N}\left(r,\frac{1}{f-a_j}\right)+\overline{N}\left(r,\frac{1}{a_1-a_3}\right)+\overline{N}\left(r,\frac{1}{a_1-a_2}\right)+\overline{N}\left(r,\frac{1}{a_2-a_3}\right)+S(r,g)\\&\leqslant\sum_{j=1}^{3}\overline{N}\left(r,\frac{1}{f-a_j}\right)+2T(r,a_3)+S(r,f).\end{aligned}\tag{7}$$

On the other hand,

$$\begin{aligned}T(r,g)&\geqslant T\left(r,\frac{f-a_1}{f-a_2}\right)-T\left(r,\frac{a_3-a_2}{a_3-a_1}\right)+O(1)=T\left(r,1+\frac{a_2-a_1}{f-a_2}\right)-T\left(r,1+\frac{a_1-a_2}{a_3-a_1}\right)+O(1)\\&=T(r,f)-T(r,a_3)+S(r,f).\end{aligned}\tag{8}$$

By combining inequalities (7) and (8), it follows that

$$T(r,f)\leqslant\sum_{j=1}^{3}\overline{N}\left(r,\frac{1}{f-a_j}\right)+3T(r,a_3)+S(r,f),$$

from which the assertion follows by using (6). □

The next result shows that 2 CM in Theorem F can be replaced with 1 CM + 1 IM by strengthening the deficiency condition (5).

Theorem 6. *Let f be a meromorphic function of finite order, let $c\in\mathbb{C}$, and let $a_1,a_2,a_3\in\widehat{\mathcal{S}}(f)$ be three distinct periodic functions with period c. If $f(z)$ and $f(z+c)$ share a_1 CM and a_3 IM, and if*

$$\limsup_{r\to\infty}\frac{N(r,\frac{1}{f-a_2})}{T(r,f)}<\frac{1}{10}, \tag{9}$$

then $f(z)=f(z+c)$ for all $z\in\mathbb{C}$.

Proof. (1) Suppose first that $a_1,a_2,a_3\in\mathcal{S}(f)$, and let $g(z)$ be as in (4). Then $g(z)$ and $g(z+c)$ share 0 CM and 1 IM, and there exists an $\gamma\in[0,1/10)$ such that

$$N(r,g) < \gamma T(r,g). \tag{10}$$

It suffices to show that $g(z) = g(z+c)$ for all $z \in \mathbb{C}$.

Suppose on the contrary that $g(z) \not\equiv g(z+c)$, and head for a contradiction. We may write

$$\frac{g(z+c)}{g(z)} = \psi(z) \quad \text{and} \quad \frac{g(z+c)-1}{g(z)-1} = \phi(z), \tag{11}$$

where ψ and ϕ are well-defined meromorphic functions of finite order satisfying $m(r,\psi) = S(r,g)$ and $m(r,\phi) = S(r,g)$ by Theorem D.

By a simple geometric observation, Lemma G and (10), we conclude that

$$N\big(r, g(z+c)\big) \leqslant N\big(r+|c|, g\big) = N(r,g) + S(r,g) < \gamma T(r,g) + S(r,g). \tag{12}$$

Since $g(z)$ and $g(z+c)$ share 0 CM, all poles of ψ are among the poles of $g(z+c)$. It now follows by (12) that

$$T(r,\psi) = N(r,\psi) + S(r,g) < \gamma T(r,g) + S(r,g). \tag{13}$$

Combining the equations in (11), we may write $\phi(z) = (\psi(z)g(z)-1)/(g(z)-1)$, from which

$$g(z) = \frac{\phi(z)-1}{\phi(z)-\psi(z)} = \left(\frac{\phi(z)-1}{\psi(z)-1} - 1\right)^{-1} + 1.$$

This gives $T(r,g) \leqslant T(r,\phi) + T(r,\psi) + O(1)$. Using (13), we obtain

$$T(r,\phi) \geqslant T(r,g) - T(r,\psi) + O(1) \geqslant (1-\gamma)T(r,g) + S(r,g). \tag{14}$$

Since $g(z)$ and $g(z+c)$ share 1 IM, it follows that the 1-points of $g(z)$ and of $g(z+c)$ are among the 1-points of ψ. Hence, by (13), we have

$$\overline{N}\left(r, \frac{1}{g-1}\right) + \overline{N}\left(r, \frac{1}{g(z+c)-1}\right) \leqslant 2\overline{N}\left(r, \frac{1}{\psi-1}\right) \leqslant 2T(r,\psi) + O(1) \leqslant 2\gamma T(r,g) + S(r,g). \tag{15}$$

From (11), we observe that the zeros and poles of ϕ are among the 1-points of either $g(z)$ or $g(z+c)$ and the poles of either $g(z)$ or $g(z+c)$. Therefore, by (10), (12) and (15), it follows that

$$\overline{N}(r,\phi) + \overline{N}\left(r, \frac{1}{\phi}\right) \leqslant 4\gamma T(r,g) + S(r,g).$$

Moreover,

$$T(r,\psi) \leqslant \frac{\gamma}{1-\gamma} T(r,\phi) + S(r,\phi)$$

by (13) and (14). Let $\varepsilon > 0$. Then, by Theorem 5 and (14), we conclude that

$$\begin{aligned}
\left(1 - \frac{3\gamma}{1-\gamma} - \varepsilon\right) T(r,\phi) &\leqslant \overline{N}(r,\phi) + \overline{N}\left(r, \frac{1}{\phi}\right) + \overline{N}\left(r, \frac{1}{\phi-\psi}\right) + S(r,\phi) \\
&\leqslant \overline{N}\left(r, \frac{g-1}{\psi-1}\right) + 4\gamma T(r,g) + S(r,\phi) \\
&\leqslant \overline{N}(r,g) + T(r,\psi) + 4\gamma T(r,g) + S(r,\phi) \\
&\leqslant \frac{6\gamma}{1-\gamma} T(r,\phi) + S(r,\phi),
\end{aligned}$$

which contradicts with the fact that $\gamma \in [0, 1/10)$.

(2) The cases when exactly one of the functions a_1, a_2, a_3 is equal to ∞ are dealt with as in part (2) of the proof of Theorem 2. □

Let f be a meromorphic function, and let $a \in \mathcal{S}(f)$. Then $n_2(r, \frac{1}{f-a})$ is the number of zeros of $f-a$ in $|z| \leqslant r$, where the simple zeros are counted once and the multiple zeros twice. Further, $n_2(r,f)$ is the number of poles of f, where the simple poles are counted once and the multiple poles twice. The corresponding integrated counting functions $N_2(r, \frac{1}{f-a})$ and $N_2(r,f)$ are defined in the usual way. Then

$$\overline{N}\left(r, \frac{1}{f-a}\right) \leqslant N_2\left(r, \frac{1}{f-a}\right) \leqslant 2\overline{N}\left(r, \frac{1}{f-a}\right) \tag{16}$$

and

$$\overline{N}(r,f) \leqslant N_2(r,f) \leqslant 2\overline{N}(r,f), \tag{17}$$

see [19, p. 365].

Theorem 7. *Let f be a meromorphic function of finite order, let $c \in \mathbb{C}$, and let $a_1, a_2, a_3 \in \widehat{\mathcal{S}}(f)$ be three distinct periodic functions with period c. If $f(z)$ and $f(z+c)$ share a_3 CM, and if*

$$\limsup_{r\to\infty} \frac{N_2(r, \frac{1}{f-a_1}) + N_2(r, \frac{1}{f-a_2})}{T(r,f)} < \frac{1}{2}, \tag{18}$$

then $f(z) = f(z+c)$ or $f(z) = f(z+2c)$ for all $z \in \mathbb{C}$.

Proof. Similarly as above, we may suppose that $a_1, a_2, a_3 \in \mathcal{S}(f)$. Let $g(z)$ be as in (4). Then $g(z)$ and $g(z+c)$ share 1 CM, and, by (18), there exist constants $\varepsilon > 0$ and $r_\varepsilon > 0$ such that

$$N_2(r, g) + N_2\left(r, \frac{1}{g}\right) < \left(\frac{1}{2} - \varepsilon\right) T(r, g), \quad r \geqslant r_\varepsilon. \tag{19}$$

By a simple geometric observation and Lemma G, we have

$$N_2\big(r, g(z+c)\big) + N_2\left(r, \frac{1}{g(z+c)}\right) \leqslant N_2\big(r+|c|, g\big) + N_2\left(r+|c|, \frac{1}{g}\right) = N_2(r, g) + N_2\left(r, \frac{1}{g}\right) + S(r, g)$$

outside of a possible exceptional set E of finite logarithmic measure. Denote $F = \mathbb{R}_+ \setminus (E \cup [0, r_\varepsilon])$, where r_ε is the constant in (19). Then F is of infinite logarithmic measure, and clearly of infinite linear measure. Hence, by (19), we get

$$N_2\big(r, g(z+c)\big) + N_2\left(r, \frac{1}{g(z+c)}\right) < \frac{1}{2} T(r, g), \quad r \in F. \tag{20}$$

By combining (19) and (20) with [19, Theorem 7.10], it follows that either $g(z) \equiv g(z+c)$ or $g(z)g(z+c) \equiv 1$. The latter possibility yields $g(z) \equiv g(z+2c)$. Therefore either $f(z) \equiv f(z+c)$ or $f(z) \equiv f(z+2c)$. $\square$

We note that if the deficiency condition (18) is replaced with

$$\limsup_{r\to\infty} \frac{\overline{N}(r, \frac{1}{f-a_1}) + \overline{N}(r, \frac{1}{f-a_2})}{T(r,f)} < \frac{1}{4},$$

and if all the other assumptions of Theorem 7 are valid, then $f(z) = f(z+c)$ or $f(z) = f(z+2c)$ for all $z \in \mathbb{C}$. This follows by (16) and (17).

Theorem H. *(See [1, Folgerung 4.1].) Let f_1 and f_2 be meromorphic functions such that*

$$\overline{N}(r, f_j) + \overline{N}\left(r, \frac{1}{f_j}\right) = S(r, f_j), \quad j = 1, 2.$$

If f_1 and f_2 share 1 IM, then $f_1(z) = f_2(z)$ or $f_1(z) f_2(z) = 1$ for all $z \in \mathbb{C}$.

Finally we introduce a deficiency condition which, together with 1 IM, forces f to be a periodic function.

Theorem 8. *Let f be a meromorphic function of finite order, let $c \in \mathbb{C}$, and let $a_1, a_2, a_3 \in \widehat{\mathcal{S}}(f)$ be three distinct periodic functions with period c. If $f(z)$ and $f(z+c)$ share a_3 IM, and if*

$$\overline{N}\left(r, \frac{1}{f-a_1}\right) + \overline{N}\left(r, \frac{1}{f-a_2}\right) = S(r, f),$$

then $f(z) = f(z+c)$ or $f(z) = f(z+2c)$ for all $z \in \mathbb{C}$.

Proof. Similarly as above, we may suppose that $a_1, a_2, a_3 \in \mathcal{S}(f)$. Let $g(z)$ be as in (4). Then $g(z)$ and $g(z+c)$ share 1 IM, and

$$\overline{N}(r, g) + \overline{N}\left(r, \frac{1}{g}\right) = S(r, g).$$

Hence, by a simple geometric observation and Lemma G, we have

$$\begin{aligned} \overline{N}\big(r, g(z+c)\big) + \overline{N}\left(r, \frac{1}{g(z+c)}\right) &\leqslant \overline{N}\big(r+|c|, g\big) + \overline{N}\left(r+|c|, \frac{1}{g}\right) \\ &= \overline{N}(r, g) + \overline{N}\left(r, \frac{1}{g}\right) + S(r, g) \\ &= S(r, g) \leqslant \varepsilon(r) T(r, g), \end{aligned}$$

where $\lim_{r\to\infty}\varepsilon(r)=0$. The second main theorem and the assumptions give

$$\begin{aligned}T(r,g)&\leqslant \overline{N}(r,g)+\overline{N}\left(r,\frac{1}{g}\right)+\overline{N}\left(r,\frac{1}{g-1}\right)+S(r,g)\\&=\overline{N}\left(r,\frac{1}{g(z+c)-1}\right)+S(r,g)\\&\leqslant T\big(r,g(z+c)\big)+S(r,g),\end{aligned}$$

that is, $(1+o(1))T(r,g)\leqslant T(r,g(z+c))$. We have proved that

$$\overline{N}\big(r,g(z+c)\big)+\overline{N}\left(r,\frac{1}{g(z+c)}\right)=S\big(r,g(z+c)\big).$$

Now Theorem H implies that either $g(z)\equiv g(z+c)$ or $g(z)g(z+c)\equiv 1$. The latter possibility yields $g(z)\equiv g(z+2c)$. Therefore either $f(z)\equiv f(z+c)$ or $f(z)\equiv f(z+2c)$. □

Theorem 8 has the following immediate consequence related to Corollary 3.

Corollary 9. *Let f be an entire function of finite order, let $c\in\mathbb{C}$, and let $a,b\in\mathcal{S}(f)$ be two distinct periodic functions with period c. If $f(z)$ and $f(z+c)$ share a IM, and if*

$$\overline{N}\left(r,\frac{1}{f-b}\right)=S(r,f),$$

then $f(z)=f(z+c)$ or $f(z)=f(z+2c)$ for all $z\in\mathbb{C}$.

4. An improved criterion for elliptic functions

Using Theorem 2, we offer the following improvement of Theorem C.

Theorem 10. *Let f be a meromorphic function, let $c_1,c_2\in\mathbb{C}$ be linearly independent over the real numbers, and let $a_1,a_2,a_3\in\widehat{\mathbb{C}}$ be three distinct values. If $f(z)$, $f(z+c_1)$ and $f(z+c_2)$ share a_1,a_2 CM and a_3 IM, then $f(z)$ is an elliptic function with periods c_1 and c_2.*

Proof. We may assume that $a_1=0, a_2=\infty$ and $a_3=1$, for otherwise we can replace f with $g=\frac{f-a_1}{f-a_2}\cdot\frac{a_3-a_2}{a_3-a_1}$. If there exists a point z_0 such that $f(z_0)=a_j$, $j=1,2,3$, then $f(z_0+kc_1)=a_j$ and $f(z_0+kc_2)=a_j$ for all $k\in\mathbb{Z}$. Therefore the parallelogram defined by the vertice points

$$lc_1+lc_2,\quad (l+1)c_1+lc_2,\quad lc_1+(l+1)c_2,\quad (l+1)c_1+(l+1)c_2$$

has the same finite number of a_j-points of $f(z)$ for all $l\in\mathbb{Z}$. Hence there exists a constant $C>0$, not depending on r, such that

$$\overline{N}\left(r,\frac{1}{f-a_j}\right)\leqslant Cr^2,\quad j=1,2,3,$$

for all $r>0$. The second main theorem yields that f is of finite order. The conclusion follows by Theorem 2. □

5. Alternative improvements of Theorem B

We proceed to find alternative improvements of Theorem B by means of Nevanlinna theory for exact differences [9]. We begin by reviewing some basic definitions and fundamental results of this theory. A more detailed presentation can be found in [9].

Let f be a meromorphic function, and let $c\in\mathbb{C}$. If $a\in\mathbb{C}$, then $n_c(r,\frac{1}{f-a})$ is the number of points z_0, $|z_0|\leqslant r$, where $f(z_0)=a$ and $f(z_0+c)=a$, counted according to the number of equal terms in the beginning of Taylor series expansions of $f(z)-a$ and $f(z+c)-a$ in a neighborhood of z_0. Such points are called c-separated a-pairs of f in the disc $\{z\colon |z|\leqslant r\}$. Further, $n_c(r,f)$ is the number of c-separated pole pairs of f, which are exactly the c-separated 0-pairs of $1/f$. The corresponding integrated counting functions $N_c(r,\frac{1}{f-a})$ and $N_c(r,f)$ are defined in the usual way. Following [9], we also define

$$\tilde{N}_c\left(r,\frac{1}{f-a}\right):=N\left(r,\frac{1}{f-a}\right)-N_c\left(r,\frac{1}{f-a}\right),$$

which counts the number of those a-points, $a\in\widehat{\mathbb{C}}$, of f which are not in c-separated pairs.

The point a is an exceptional paired value of f with the separation c if the following property holds for all a-points of f: Whenever $f(z)=a$ then also $f(z+c)=a$ with the same or higher multiplicity.

Theorem I. *(See [9, Theorem 2.5].) Let $c \in \mathbb{C}$, and let f be a meromorphic function of finite order such that $f(z) - f(z+c) \not\equiv 0$. Let $q \geqslant 2$, and let $a_1, a_2, \ldots, a_q \in \widehat{\mathcal{S}}(f)$ be distinct periodic functions with period c. Then*

$$(q-1)T(r,f) \leqslant \tilde{N}_c(r,f) + \sum_{k=1}^{q} \tilde{N}_c\left(r, \frac{1}{f-a_k}\right) + S(r,f). \tag{21}$$

Theorem I has several neat consequences including a c-separated analogue of the classical defect relation [9, Corollary 2.6]. The shortest proof for Picard's theorem is by means of the usual second fundamental theorem by Nevanlinna. The following analogue of Picard's theorem [9, Corollary 2.7] follows immediately by Theorem I: If a finite order meromorphic function f has three exceptional paired values with the separation c, then f is a periodic function with period c.

Let f be a meromorphic function of order $\rho(f) \in (0,\infty]$. We say that a is a Borel exceptional paired value of f with the separation c if

$$\limsup_{r\to\infty} \frac{\log^+ \tilde{N}_c(r, \frac{1}{f-a})}{\log r} < \rho(f).$$

Note that if a is an exceptional paired value of f with the separation c, then $\tilde{N}_c(r, \frac{1}{f-a}) = O(1)$, and hence a is also a Borel exceptional paired value of f with the separation c. The following analogue of Borel's theorem is an immediate consequence of Theorem I.

Corollary 11. *Let $c \in \mathbb{C} \setminus \{0\}$. If a meromorphic function f of order $\rho(f) \in (0,\infty)$ has three Borel exceptional paired values with the separation c, then f is a periodic function with period c.*

Since $\tilde{N}_c(r, \frac{1}{f-a}) \leqslant N(r, \frac{1}{f-a})$ holds for any meromorphic f and any $a \in \widehat{\mathcal{S}}(f)$, we see that the following result is a slight improvement of Theorem F.

Theorem 12. *Let f be a meromorphic function of finite order, let $c \in \mathbb{C}$, and let $a_1, a_2, a_3 \in \widehat{\mathcal{S}}(f)$ be three distinct periodic functions with period c. If $f(z)$ and $f(z+c)$ share a_1 and a_3 CM, and if*

$$\limsup_{r\to\infty} \frac{\tilde{N}_c(r, \frac{1}{f-a_2})}{T(r,f)} < 1,$$

then $f(z) = f(z+c)$ for all $z \in \mathbb{C}$.

The conclusion of Theorem 12 follows by Theorem I and the following lemma.

Lemma 13. *Let f be a meromorphic function, let $c \in \mathbb{C}$, and let $a \in \widehat{\mathcal{S}}(f)$ be a periodic function with period c. If f shares a CM with $f(z+c)$, then 0 is an exceptional paired value of $f - a$ with the separation c, and*

$$\tilde{N}_c\left(r, \frac{1}{f-a}\right) \leqslant 0.$$

Proof. If 0 is a Picard value of $f(z) - a(z)$, then 0 is also a Picard value of $f(z+c) - a(z)$, and the conclusions follow trivially. Suppose then that 0 is not a Picard value of $f - a$. Let z_0 be such that $f(z_0) - a(z_0) = 0$ with multiplicity p. Since $f(z)$ shares a CM with $f(z+c)$, it follows that $f(z_0 + kc) - a(z_0) = 0$ with the same multiplicity p for all $k \in \mathbb{Z}$. The conclusions now follow by definition. $\square$

We note that the combination of Theorem I and Lemma 13 also yields an alternative proof for Theorem B.

Theorem 14. *Let f be a meromorphic function of finite order, let $c \in \mathbb{C}$, and let $a_1, a_2, a_3 \in \widehat{\mathcal{S}}(f)$ be three distinct periodic functions with period c. Given $\varepsilon \in (0, \frac{1}{3})$, if any zero of $f(z) - a_j(z)$ $(j = 1,2,3)$, with multiplicity p, is a zero of $f(z+c) - a_j(z)$, with multiplicity $q > \frac{2-(1-\varepsilon)\varepsilon}{3} p + \varepsilon$, then $f(z) = f(z+c)$ for all $z \in \mathbb{C}$.*

Theorem 14 is closely related to the generally open 3 IM situation. Note that if $f(z)$ and $f(z+c)$ share three values CM, then $p = q$ in Theorem 14, which therefore is an improvement of Theorem B. Also note that $\frac{2-(1-\varepsilon)\varepsilon}{3} + \varepsilon < 1$ for $\varepsilon \in (0, \frac{1}{3})$, and hence the restriction for the multiplicity q is automatically satisfied in the case $p = 1$.

Proof of Theorem 14. By the value sharing assumptions, all zeros of $f - a_j$ are in c-separated pairs for $j = 1,2,3$. Clearly

$$N_c\left(r, \frac{1}{f-a_j}\right) > \frac{2-(1-\varepsilon)\varepsilon}{3} N\left(r, \frac{1}{f-a_j}\right) + \varepsilon \overline{N}\left(r, \frac{1}{f-a_j}\right), \quad j = 1,2,3,$$

and so

$$\tilde{N}_c\left(r, \frac{1}{f-a_j}\right) < \frac{1+(1-\varepsilon)\varepsilon}{3} N\left(r, \frac{1}{f-a_j}\right) - \varepsilon \overline{N}\left(r, \frac{1}{f-a_j}\right), \quad j=1,2,3.$$

Suppose on the contrary that $f(z) \not\equiv f(z+c)$. Then, by Theorem I, we have

$$T(r,f) \leqslant \sum_{j=1}^{3} \tilde{N}_c\left(r, \frac{1}{f-a_j}\right) + S(r,f) \leqslant \bigl(1+(1-\varepsilon)\varepsilon\bigr) T(r,f) - \varepsilon \sum_{j=1}^{3} \overline{N}\left(r, \frac{1}{f-a_j}\right) + S(r,f),$$

and hence

$$\sum_{j=1}^{3} \overline{N}\left(r, \frac{1}{f-a_j}\right) \leqslant (1-\varepsilon) T(r,f) + S(r,f). \tag{22}$$

Combining (22) with [12, Theorem 2.5], we obtain

$$\bigl(1+o(1)\bigr) T(r,f) \leqslant (1-\varepsilon) T(r,f),$$

which is a contradiction. Hence $f(z) = f(z+c)$ for all $z \in \mathbb{C}$. □

An analogous reasoning yields the next result for entire functions offering an improvement of [13, Corollary 2.2(a)].

Theorem 15. *Let f be an entire function of finite order, let $c \in \mathbb{C}$, and let $a_1, a_2 \in \mathcal{S}(f)$ be two distinct periodic functions with period c. Given $\varepsilon \in (0, \frac{1}{2})$, if any zero of $f(z) - a_j(z)$ $(j = 1, 2)$, with multiplicity p, is a zero of $f(z+c) - a_j(z)$, with multiplicity $q > \frac{1-(1-\varepsilon)\varepsilon}{2} p + \varepsilon$, then $f(z) = f(z+c)$ for all $z \in \mathbb{C}$.*

6. Functions f and $f \circ p$ share values

The next result is a meromorphic analogue of Theorem A.

Theorem 16. *Let f be a transcendental meromorphic function and p an entire function. If f and $f \circ p$ share three distinct values $a_1, a_2, a_3 \in \widehat{\mathbb{C}}$ CM, then $p(z) = \alpha z + \beta$ for some constants $\alpha, \beta \in \mathbb{C}$ with $\alpha^n = 1$ for some $n \in \mathbb{N}$. The same conclusion holds if p is of finite order and CM is replaced with IM.*

To prove Theorem 16, we need the following lemma.

Lemma 17. *Let f be a meromorphic function and $p(z) = \alpha z + \beta$, where $\alpha, \beta \in \mathbb{C} \setminus \{0\}$ and $|\alpha| = 1$. If f and $f \circ p$ share $a \in \widehat{\mathbb{C}}$ IM, then there exists a positive integer n such that $\alpha^n = 1$, or f takes the value a at most once.*

Proof. Suppose that f takes the value a at least twice, and that $\alpha \neq 1$. Let $z_0 \neq \beta/(1-\alpha)$ be such that $f(z_0) = f(\alpha z_0 + \beta) = a$. Since z_0 is not the fixed point of p, the value

$$z_n = \alpha^n z_0 + \beta \frac{1-\alpha^n}{1-\alpha}$$

is an a-point of f for each positive integer n. Since $|\alpha| = 1$ by the assumption, we obtain

$$|z_n| \leqslant |z_0| + \frac{2|\beta|}{|1-\alpha|},$$

which means that either the a-point sequence $\{z_n\}$ must accumulate to a finite value or $z_m = z_p$ for some distinct positive integers m and p. The former is impossible while the latter easily yields $\alpha^{m-p} = 1$. □

Proof of Theorem 16. Without loss of generality, we may assume that the shared values are 0, 1 and ∞. By the second main theorem,

$$T(r,f) \leqslant 3T(r, f \circ p) + S(r,f), \tag{23}$$

$$T(r, f \circ p) \leqslant 3T(r,f) + S(r, f \circ p), \tag{24}$$

$$T(r, f \circ p \circ p) \leqslant 3T(r,f) + S(r, f \circ p \circ p), \tag{25}$$

and hence $\mathcal{S}(f) = \mathcal{S}(f \circ p) = \mathcal{S}(f \circ p \circ p)$. By the value sharing assumption, there exist entire functions a and b such that

$$\frac{f\circ p}{f}=e^a \quad \text{and} \quad \frac{f\circ p-1}{f-1}=e^b. \tag{26}$$

Therefore

$$\frac{f\circ p\circ p}{e^a f}=e^{a\circ p} \quad \text{and} \quad \frac{f\circ p\circ p-1}{e^b(f-1)}=e^{b\circ p}. \tag{27}$$

We proceed to show that p must be a polynomial. Assume on the contrary that p is transcendental. We make two observations:

(1) If a is a constant, then clearly $T(r,e^a)=S(r,f)$. Suppose then that a is non-constant. Now, by [4, Theorem 1(ii)], for an arbitrarily large constant $M>1$ there exists a constant $r_M>0$ such that

$$T\big(r,e^{a\circ p}\big)\geqslant MT\big(r,e^a\big), \quad r\geqslant r_M. \tag{28}$$

By using (28), (27), (25) and elementary Nevanlinna theory, we obtain

$$\begin{aligned} MT\big(r,e^a\big) &\leqslant T(r,f\circ p\circ p)+T\big(r,e^a\big)+T(r,f)+O(1)\\ &\leqslant 4T(r,f)+T\big(r,e^a\big)+S(r,f). \end{aligned}$$

Hence, for all r outside of a possible exceptional set of finite linear measure, we have the estimate

$$\frac{T(r,e^a)}{T(r,f)}\leqslant\frac{5}{M-1}.$$

Since $M>1$ is arbitrarily large, it follows that $T(r,e^a)=S(r,f)$.

(2) If b is either a constant or non-constant, it follows similarly as in (1) that $T(r,e^b)=S(r,f)$ always holds.

We have shown that $T(r,e^a)=S(r,f)$ and $T(r,e^b)=S(r,f)$ always hold, provided that p is transcendental. Combining the equations in (26) gives

$$\big(e^a-e^b\big)f=1-e^b,$$

which results in $e^a\equiv e^b\equiv 1$. As a consequence, $f\circ p=f$. By [4, Theorem 2(ii)],

$$\limsup_{r\to\infty}\frac{T(r,f\circ p)}{T(r,f)}=\infty,$$

which is a contradiction. Therefore p must be a polynomial. Similarly, as in the proof of [13, Theorem 1.5], we see that $p(z)=\alpha z+\beta$, where $\alpha,\beta\in\mathbb{C}$ and $|\alpha|=1$. Since f is transcendental, it takes at least one of the shared values a_1,a_2,a_3 infinitely many times. Hence, by Lemma 17, there exists a positive integer n such that $\alpha^n=1$.

Finally, we suppose that f shares the values 0, 1 and ∞ IM with $f\circ p$, where p is a non-constant entire function of finite order. We may write

$$\frac{f\circ p}{f}=\psi \quad \text{and} \quad \frac{f\circ p-1}{f-1}=\phi,$$

where ψ and ϕ are well-defined meromorphic functions. If ψ is a constant, then clearly $T(r,\psi)=S(r,f)$. Supposing that p is transcendental (yet of finite order), we use [4, Theorem 3(i)] to conclude that, for an arbitrarily large constant $M>1$, there exists a constant $r_M>0$ such that

$$T(r,\psi\circ p)\geqslant MT(r,\psi), \quad r\geqslant r_M.$$

This corresponds to the estimate in (28), and leads to $T(r,\psi)=S(r,f)$ just as in case (1) above. By replacing e^a with ψ and e^b with ϕ, the rest of the proof follows that of the CM-case above, word for word. □

Theorem 18. *Let f be an entire function having a Picard value $a\in\mathbb{C}$, and let p be an entire function. If f and $f\circ p$ share a value $b\in\mathbb{C}\setminus\{a\}$ IM, then one of the following assertions hold*:

(1) *$f\equiv f\circ p$ and $p(z)=\alpha z+\beta$ for some $\alpha,\beta\in\mathbb{C}$ such that $\alpha^n=1$ for some $n\in\mathbb{N}$.*
(2) *$f\equiv f\circ p\circ p$ and $p(z)=\alpha z+\beta$ for some $\alpha,\beta\in\mathbb{C}$ such that $\alpha^{2n}=1$ for some $n\in\mathbb{N}$.*

Proof. Define $g=\frac{f-a}{b-a}$. Then g and $g\circ p$ are both entire functions having zero as a Picard value. Moreover, g and $g\circ p$ share 1 IM. By Theorem H we conclude that either $g\equiv g\circ p$ or $g\equiv\frac{1}{g\circ p}$. The latter possibility yields $g\equiv g\circ p\circ p$. Therefore either $f\equiv f\circ p$ or $f\equiv f\circ p\circ p$.

We proceed to prove the assertions on p. Suppose first that $f \equiv f \circ p$. By Theorem 16 we deduce that $p(z) = \alpha z + \beta$ for some $\alpha, \beta \in \mathbb{C}$ such that $\alpha^n = 1$ for some $n \in \mathbb{N}$. Suppose then that $f \equiv f \circ p \circ p$. We may apply Theorem 16, with $p \circ p$ in place of p, to deduce that $(p \circ p)(z) = Az + B$ for some $A, B \in \mathbb{C}$ such that $A^n = 1$ for some $n \in \mathbb{N}$. Consequently, $p(z) = \alpha z + \beta$ for some $\alpha, \beta \in \mathbb{C}$ such that $\alpha^{2n} = 1$ for some $n \in \mathbb{N}$. □

Corollary 19. *Let f be a meromorphic function and p an entire function, and let $a_1, a_2, a_3, a_4 \in \widehat{\mathbb{C}}$ be pairwise different. If f and $f \circ p$ share a_1, a_2 CM and a_3, a_4 IM, then $f \equiv f \circ p$ or $f \equiv f \circ p \circ p$.*

Proof. By [7, Theorem 1], the functions f and $f \circ p$ share all four values $a_1, \ldots, a_4$ CM. Then, by the classical 4-point theorem, see [16, p. 18], we may assume that a_3 and a_4 are Picard values of f and of $f \circ p$. Hence $g = \frac{f-a_3}{f-a_4}$ is an entire function and avoids the value zero. Moreover, g and $g \circ p$ share the values $\frac{a_j-a_3}{a_j-a_4}$, $j = 1, 2$, CM. Hence, by Theorem 18, we obtain the desired conclusion. □

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